

# RINGS AND MODULES



*POTENT RING*



NILPOTENT RING

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These notes were prepared for students at Macquarie University in Australia but are freely available to anyone. However if you make use of them and are not a Macquarie University student it would be nice if you could email me at [christopherdonaldcooper@gmail.com](mailto:christopherdonaldcooper@gmail.com) to let me know where you are from. And, if you are from outside of Australia perhaps you could send me a postcard of where you are from to pin up on my wall (Christopher Cooper, 31 Epping Avenue, EASTWOOD, NSW 2122, Australia).



# INTRODUCTION

These notes provide a fundamental training in Ring Theory. The first chapter lays the general foundations, and the second chapter deals with an important class of commutative rings.

An *algebra* is simply a ring that also has the structure of a vector space over some field. Most rings of any importance are algebras so, for the most part, the theory of rings and the theory of algebras are essentially the same. The advantage of studying algebras is that, from time to time, we can make use of the vector space structure.

However, from the third chapter the focus is on **non-commutative rings**. The goal is to prove the **Wedderburn Structure Theorem** for ‘semi-simple algebras with descending chain condition on right ideals over  $\mathbb{C}$ ’. Never mind what these conditions are at this stage. The important fact, as far as Representation Theory is concerned, is that group rings of finite groups satisfy these properties and so we can say something about the structure of group rings, and this in turn leads to the Fundamental Theorem of Representation Theory.

And what is a **group ring**? Groups have only a single operation, that we will represent as multiplication. We can multiply permutations, but we can’t add them, or can we? In fact we *can* add permutations, or indeed any elements in any multiplicative group, simply by adding

them *formally*. That is, we take formal linear combinations of the elements, such as, starting with  $S_3$ :

$$3(123) - 5(12) + \frac{1}{2}(23).$$

If we were to be asked to simplify this expression we would have to say, “it’s already in its simplest form”. If we added this to a similar expression we would combine ‘like terms’ so that every element in the group ring of  $S_3$  could be written as a formal sum of the form:

$$aI + b(123) + c(132) + d(12) + e(13) + f(23).$$

Multiplying two such expressions we simply multiply them term by term, multiplying the coefficients in the field and the group elements in the group.

The aim of these notes is not to get to the Wedderburn Structure Theorem as quickly as possible, but rather to take a rather more leisurely and circuitous tour of radicals in rings, following to some extent the route laid down by Divinsky in his book *Rings and Radicals*. After discussing radicals in general, and some specific examples in particular, we then concentrate on nil-semisimplicity. Likewise we investigate the Ascending Chain Condition as well as the Descending one, before focussing on the Descending Chain condition that is needed for Wedderburn’s theorem.

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